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THE REDUCTION OF TRUNCATION ERROR BY EXTRAPOLATION TECHNIQUES [1]

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ABSTRACT

By the use of two separate mesh sizes instead of one in computing finite differences, an extrapolation to effective "zero" mesh size may be made in order to reduce the truncation error, as originally suggested by Richardson. This technique of "difference extrapolation" is applied to the estimation of individual differentials in the barotropic vorticity equation, and here corresponds to the use of "second-order" finite differences. The truncation-induced phase speed lag of the difference solution relative to the true solution is shown to be systematically reduced, especially for the shorter waves. Next, the extrapolation is applied with two separate solutions of the barotropic difference equation, with the result that the phase speeds are further improved, but at the expense of an amplitude distortion of about 10 percent. This amplitude distortion may be removed for a particular wavelength, and a small further phase speed improvement obtained, but the amplitude distortion remains for other wavelengths. These methods of "solution extrapolation" are therefore felt to be unsuitable for routine use. The method of "difference extrapolation," however, preserves the solution's amplitude, and if used in conjunction with a suitable smoothing procedure should result in a net error reduction for those waves resolved by the mesh and retained by the smoothing.

1. INTRODUCTION

In the approximation of spatial derivatives by finite differences, a truncation error is made which in general depends upon the size of the finite space increment, as well as upon the wavelength and orientation of the continuous field being estimated. The most widely-used procedure is the familiar centered space difference, which we may illustrate in the case of the first derivative of a (continuous) function f as

$$\frac{\partial f}{\partial x} \simeq \Delta f \equiv \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}, \tag{1}$$

where x is a typical space coordinate and Δx is the mesh size. The error of this approximation is easily shown to be of the order $(\Delta x)^2$, i.e.,

$$e(f) \equiv \Delta f - \frac{\partial f}{\partial x} = O(\Delta x^2). \tag{2}$$

This approximation is furthermore a consistent one in the sense that $e(f) \rightarrow 0$ as $\Delta x \rightarrow 0$. It is the purpose of this study to utilize such Δx-dependence of the truncation error in order to improve the accuracy of the finite-difference approximation itself, using an extrapolation technique. The procedure to be described will be seen to be related to several proposed methods for the reduction of truncation error and for smoothing.

2. THE EXTRAPOLATION TECHNIQUE

Since the accuracy of the difference approximation depends upon the mesh spacing, it occurs to one that from, say, *two* finite difference estimates, made with different grid increments, an improved estimate of the derivative might be made. Considering the estimation of $\partial f/\partial x$ for

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example, and denoting the two grid increments by Δx_1, and Δx_2, we have

$$(\Delta f)_{1,2}=\frac{f(x+\Delta x_{1,2})-f(x-\Delta x_{1,2})}{2\Delta x_{1,2}}, \tag{3}$$

where $(\Delta f)_{1,2}$ denotes either of the two difference estimates of $\partial f/\partial x$. Since the error of $(\Delta f)_{1,2}$ is of the order $(\Delta x)^2$, we may extrapolate to an effective zero mesh size by forming the linear combination

$$(\Delta f)_0=\frac{\rho^2(\Delta f)_1-(\Delta f)_2}{\rho^2-1}, \tag{4}$$

where $\rho=\Delta x_2/\Delta x_1>1$. This "extrapolation to the limit" is illustrated in figure 1, and was evidently first suggested by Richardson [6].

In order to show that the estimate (4) is a systematic improvement over either $(\Delta f)_1$ or $(\Delta f)_2$ alone, we may expand f in a Taylor series with remainder about $x=0$,

$$f(x\pm\Delta x_1)=f(x)\pm\Delta x_1\frac{\partial f}{\partial x}+\frac{\Delta x_1^2}{2}\frac{\partial^2 f}{\partial x^2}\pm\frac{\Delta x_1^3}{6}\frac{\partial^3 f}{\partial x^3}+\frac{\Delta x_1^4}{24}\frac{\partial^4 f}{\partial x^4}\pm\frac{\Delta x_1^5}{120}\left(\frac{\partial^5 f}{\partial x^5}\right)_{\theta_1,\theta_2}, \tag{5}$$

$$f(x\pm\Delta x_2)=f(x\pm\rho\,\Delta x_1)=f(x)\pm\rho\,\Delta x_1\frac{\partial f}{\partial x}+\frac{\rho^2\Delta x_1^2}{2}\frac{\partial^2 f}{\partial x^2}\pm\frac{\rho^3\Delta x_1^3}{6}\frac{\partial^3 f}{\partial x^3}+\frac{\rho^4\Delta x_1^4}{24}\frac{\partial^4 f}{\partial x^4}\pm\frac{\rho^5\Delta x_1^5}{120}\left(\frac{\partial^5 f}{\partial x^5}\right)_{\theta_3,\theta_4}, \tag{6}$$

where $\theta_1, \ldots, \theta_4$ are suitable points in the vicinity of the origin. Inserting these expressions into (3) and (4), we find the truncation error

$$e(f)=(\Delta f)_0-\frac{\partial f}{\partial x}=O(\Delta x^4). \tag{7}$$

This result was only to be expected, however, since the ρ^2-extrapolation in (4) was designed to eliminate the dominant $(\Delta x)^2$ error-dependence of $(\Delta f)_{1,2}$.

This extrapolation technique has been used with some success in steady-state problems of engineering (Salvadori [8]), and is a relatively well-known procedure in the numerical analysis of linear differential equations (see, for example, Buckingham [1] or Hartree [3]). In the nonlinear, time-dependent equations typical of dynamical weather prediction, however, we have no guarantee that its use will result in a systematic improvement. In the first place, the extrapolation might be performed at each time step in the calculation of the non-homogeneous terms, or might be applied to the entire solutions from the separate grids. Moreover, in the presence of several wavelengths in the field of the dependent variable, the extrapolation may improve the difference estimates for only certain waves and fail to provide an overall error reduction of the order implied by (7). In spite of these misgivings, it seems worthwhile to give the extrapolation technique some further consideration, particularly in view of its relations to other computational schemes discussed below.

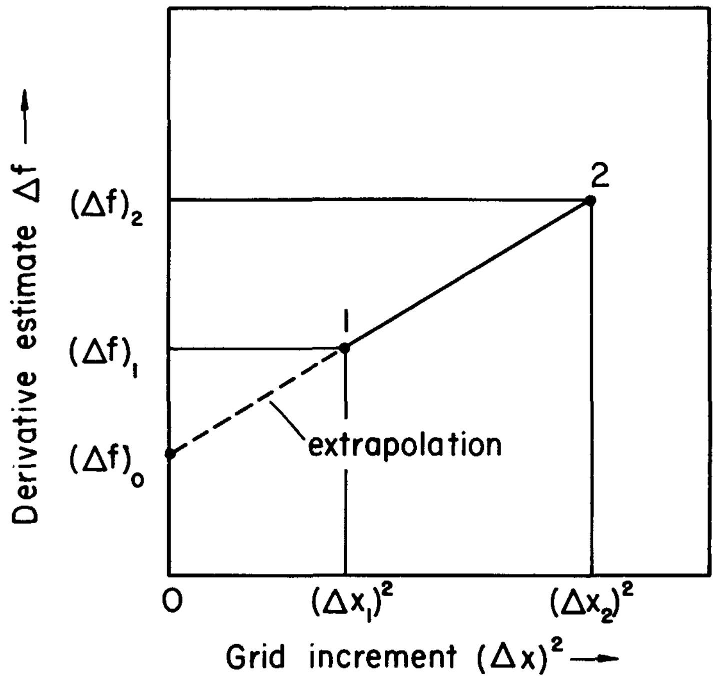

Figure 1.—The Richardson extrapolation technique for the reduction of truncation error, applied to the calculation of the derivative $\partial f/\partial x$ by centered differences.

For the sake of clarity, let us select $\Delta x_2/\Delta x_1=2$, and again consider the estimation of $\partial f/\partial x$ by centered differences. Expanding (4) we find, with $\rho^2=4$,

$$(\Delta f)_0=\frac{4}{3}(\Delta f)_1-\frac{1}{3}(\Delta f)_2, \tag{8}$$

which, upon use of (3), in turn yields

$$(\Delta f)_0=\frac{1}{12\Delta x_1}(f_{-2\Delta x}-8f_{-\Delta x}+8f_{\Delta x}-f_{2\Delta x}), \tag{9}$$

where $f_{-2\Delta x}$ denotes $f(x-2\Delta x)$, etc. This expression is recognized as just the "five-point" approximation to the first derivative.

The extrapolation technique may also be employed in the estimation of other differentials, of which the familiar Laplacian $\nabla^2 f$ is a convenient example. Proceeding as before, we find

$$(\Delta^2 f)_0=\frac{\rho^2(\Delta^2 f)_1-(\Delta^2 f)_2}{\rho^2-1}, \tag{10}$$

where $(\Delta^2 f)_{1,2}$ are the difference estimates of $\nabla^2 f$ with a truncation error $O(\Delta x^2)$. The error of the extrapolated estimate $(\Delta^2 f)_0$ is now $O(\Delta x^4)$. If we select $\rho^2=2$, we then have simply

$$(\Delta^2 f)_0=2(\Delta^2 f)_1-(\Delta^2 f)_2. \tag{11}$$

This formula is the same as that found by Knighting [4] in considering both the conventional five-point Laplacian estimate and that involving the four "corner" points at a distance $\sqrt{2}$ greater from the central point. This scheme is generated by a 45° rotation of the grid axes, and is an improvement on the directionally-averaged estimate of the Laplacian proposed by Thompson [10].

As a final case of meteorological interest, we note that the extrapolation method may be used in estimating the Jacobian $J(a, b)$ of two scalar variables a and b. The conventional estimate

$$I_1(a,b)=(4\Delta x_1^2)^{-1}[(a_{i+1,j}-a_{i-1,j})(b_{i,j+1}-b_{i,j-1}) \\ -(a_{i,j+1}-a_{i,j-1})(b_{i+1,j}-b_{i-1,j})] \quad (12)$$

may be combined with the analogous estimate with Δx_2, yielding

$$I_0(a,b)=[\rho^2 I_1(a,b)-I_2(a,b)][\rho^2-1]^{-1}, \quad (13)$$

since the dominant truncation error in each case is $O(\Delta x^2)$. The error of the estimate $I_0(a,b)$ is readily shown to be $O(\Delta x^4)$, and hence a systematic improvement over either $I_1(a,b)$ or $I_2(a,b)$ alone. If we select $\rho=2$ as before, we find

$$I_0(a,b)=2I_1(a,b)-I_2(a,b) \quad (14)$$

which has also been suggested by Knighting [4] using the 45° "rotated" axes for $I_2(a,b)$. Knighting, Jones, and Hinds [5] have used both $I_1(a,b)$ and $I_2(a,b)$ as *separate* estimates of the Jacobian in numerical integration of a simple dynamical model, and find in general that the "rotated" axes estimate ($I_2(a,b)$) yields a smoother field, and in some cases improved the prediction. From the present viewpoint it would have been interesting if *both* estimates were used as in (14).

3. THE EXTRAPOLATION OF DIFFERENTIALS IN THE BAROTROPIC VORTICITY EQUATION

Instead of the estimation of a single differential, the meteorological prediction problem involves the solution of a complete differential equation. The simplest of such equations is that of the linearized barotropic model,

$$\frac{\partial^3\psi}{\partial t\partial x^2}+U\frac{\partial^3\psi}{\partial x^3}+\beta\frac{\partial\psi}{\partial x}=0, \quad (15)$$

where ψ is the stream function of the nondivergent flow, U the assumed constant zonal current, β the Rossby parameter, and x and t the eastward spatial and time coordinates, respectively. As noted earlier, there are evidently two ways of applying the Richardson extrapolation technique: Either to improve the estimates of each differential of (15) separately, or to improve the solution by a suitable combination of two solutions found with different mesh sizes. We shall consider both methods in this and the following sections.

The first extrapolation method, which might be termed "extrapolation of differentials," is based upon the fact that the truncation error of $\partial\psi/\partial x$, $\partial^2\psi/\partial x^2$, and $\partial^3\psi/\partial x^3$ is $O(\Delta x^2)$ when the usual centered space differences are used. The use of centered time differences in the solution, however, introduces some awkwardness into the analysis by virtue of its inapplicability for the first time step. Forward time differences, on the other hand, are unstable and consequently unsuitable. There remains the method of implicit differences (Gates [2]), and it will be used here in order to simplify the analysis and at the same time to employ a method applicable in practice.

The essence of the implicit difference approximation is the use of information at *both* the current discrete time step τ and at the next time step $\tau+1$ in order to evaluate centered spatial differences. For the derivative $\partial\psi/\partial x$ we thus have

$$\frac{\partial\psi}{\partial x}\simeq\frac{1}{2}((\Delta f)_\tau+(\Delta f)_{\tau+1}), \quad (16)$$

where Δf denotes the usual centered difference estimate as in (3). If both $(\Delta f)_\tau$ and $(\Delta f)_{\tau+1}$ are now improved by the extrapolation technique with $\rho^2=\Delta x_2/\Delta x_1=4$ for convenience, we have

$$\frac{\partial\psi}{\partial x}\simeq(24\Delta x_1)^{-1}[8(\psi_{m+1,\tau+1}-\psi_{m-1,\tau+1}+\psi_{m+1,\tau}-\psi_{m-1,\tau}) \\ -(\psi_{m+2,\tau+1}-\psi_{m-2,\tau+1}+\psi_{m+2,\tau}-\psi_{m-2,\tau})], \quad (17)$$

where

$$\psi_{m,\tau}=\psi(m\Delta x,\tau\Delta t),\ \ m=0,\pm1,\ldots,\ \ \tau=0,1,\ldots.$$

In a similar manner we find the extrapolated implicit difference estimates

$$\frac{\partial^2\psi}{\partial x^2}\simeq(24\Delta x_1^2)^{-1}[16(\psi_{m+1,\tau+1}+\psi_{m-1,\tau+1}-2\psi_{m,\tau+1}+\psi_{m+1,\tau} \\ +\psi_{m-1,\tau}-2\psi_{m,\tau})-(\psi_{m+2,\tau+1}+\psi_{m-2,\tau+1} \\ -2\psi_{m,\tau+1}+\psi_{m+2,\tau}+\psi_{m-2,\tau}-2\psi_{m,\tau})], \quad (18)$$

$$\frac{\partial^3\psi}{\partial x^3}\simeq(96\Delta x_1^3)^{-1}[32(\psi_{m+2,\tau+1}-2\psi_{m+1,\tau+1}+2\psi_{m-1,\tau+1} \\ -\psi_{m-2,\tau+1}+\psi_{m-2,\tau}-2\psi_{m+1,\tau}+2\psi_{m-1,\tau}-\psi_{m-2,\tau}) \\ -(\psi_{m+4,\tau+1}-2\psi_{m+2,\tau+1}+2\psi_{m-2,\tau+1}-\psi_{m-4,\tau+1} \\ +\psi_{m+4,\tau}-2\psi_{m+2,\tau}+2\psi_{m-2,\tau}-\psi_{m-4,\tau})], \quad (19)$$

$$\frac{\partial^3\psi}{\partial t\partial x^2}\simeq(12\Delta t\Delta x_1^2)^{-1}[16(\psi_{m+1,\tau+1}+\psi_{m-1,\tau+1}-2\psi_{m,\tau+1} \\ -\psi_{m+1,\tau}-\psi_{m-1,\tau}+2\psi_{m,\tau})-(\psi_{m+2,\tau+1}+\psi_{m-2,\tau+1} \\ -2\psi_{m,\tau+1}-\psi_{m+2,\tau}-\psi_{m-2,\tau}+2\psi_{m,\tau})]. \quad (20)$$

Inserting these expressions into (15) and assuming a solution of the form $\psi_{m,\tau}=A(\tau)e^{im\alpha}$, where $\alpha=2\pi\Delta x/L$ with L the wavelength, we find after some manipulation

$$(a+ib)A(\tau+1)-(a-ib)A(\tau)=0, \quad (21)$$

where

$$a=\cos\alpha-7, \quad (22)$$

$$b=\frac{\Delta t\sin\alpha}{2\Delta x_1}\left[U(\cos^2\alpha+\cos\alpha-8)+\beta\Delta x_1^2\left(\frac{\cos\alpha-4}{\cos\alpha-1}\right)\right]. \quad (23)$$

The so-called amplification matrix of (21) is given by

$$G=\frac{a-ib}{a+ib}, \quad (24)$$

from which we see that $|G|^{1/2}=1$, a characteristic of the (stable) implicit difference scheme. The solution of (21) for arbitrary $t=\tau\Delta t$ may now be written as

$$A(\tau)=\Psi G^{\tau}, \quad (25)$$

where Ψ is the amplitude of the assumed initial condition $\psi_{m,0}=\Psi e^{im\alpha}$; i.e., $A(0)=\Psi$. From (25) we may then write the complete solution as

$$\psi_{m,\tau}=\Psi|G|^{\tau/2}\exp\{i[m\alpha+\tau\tan^{-1}(2ab/a^2-b^2)]\}. \quad (26)$$

Recalling $|G|^{1/2}=1$ and after some further manipulation, we may write this solution as

$$\psi_{m,\tau}=\Psi\exp\{i[m\alpha-\tau\tan^{-1}(4\lambda^*/4-\lambda^{*2})]\}, \quad (27)$$

where

$$\lambda^*=\frac{\Delta t}{\Delta x_1}\sin\alpha\left\{U\left[\frac{\cos^2\alpha+\cos\alpha-8}{\cos\alpha-7}\right]-\frac{\beta L^2/4\pi^2}{\left(\frac{\sin\alpha/2}{\alpha/2}\right)^2}\left[\frac{2\cos\alpha-8}{\cos\alpha-7}\right]\right\}. \quad (28)$$

Comparing λ^* with the corresponding parameter

$$\lambda=\frac{\Delta t}{\Delta x}\sin\alpha\left[U-\frac{\beta L^2/4\pi^2}{\left(\frac{\sin\alpha/2}{\alpha/2}\right)^2}\right] \quad (29)$$

occurring in the implicit difference solution of the barotropic vorticity equation without extrapolation (Gates [2]), we note that they differ only by the presence of the bracketed terms of (28). The variation of λ^* with L and Δx_1 is shown in figure 2 for the selected values $\Delta t=1$ hr., $U=20$ m. sec.$^{-1}$, and $\beta(45°\text{ lat.})=1.619\times10^{-13}$ cm.$^{-1}$ sec.$^{-1}$ By comparison with the corresponding values of λ also given, we note $\lambda^*\geq\lambda$ for all Δx and L.

The solutions with and without extrapolation may now be conveniently compared by writing (27) in the form

$$\psi_{m,\tau}=\Psi\exp\{ik[m\Delta x_1-\tau\Delta tC_N^*]\}, \quad (30)$$

where C_N^* is the phase speed of the numerical solution and $k=2\pi/L$ is the wave number. Here C_N^* is given by

$$C_N^*=\frac{L}{2\pi\Delta t}\tan^{-1}\left(\frac{4\lambda^*}{4-\lambda^{*2}}\right), \quad (31)$$

and its variation with Δx and L is shown in figure 3, along with the corresponding data for C_N without extrapolation. The phase speeds are seen to be systematically increased for all L and Δx, particularly in the intermediate cases, say $5\Delta x\leq L\leq10\Delta x$. If we compare these curves with the corresponding Rossby phase speed

$$C_R=U-\frac{\beta L^2}{4\pi^2} \quad (32)$$

of the analytic solution of (15), we may say that the phase speed error of the numerical solution is approximately halved by the use of the Richardson extrapolation technique with $\rho=2$.

The departure of the numerical phase speeds from the corresponding Rossby phase speeds of the continuous solution is due to both space and time truncation errors. The present extrapolation technique, we note, seeks to reduce only the spatially induced error, and it is therefore of interest to investigate its relative efficiency in comparison to the limiting case of $\Delta x\rightarrow0$. From (28) we first see that

$$\lim_{\Delta x\rightarrow0}\lambda^*=\frac{2\pi\Delta t}{L}C_R, \quad (33)$$

where C_R is given by (32). Hence, from (31) we have

$$\lim_{\Delta x\rightarrow0}C_N^*=\frac{L}{2\pi\Delta t}\tan^{-1}\left[\frac{2\pi\Delta t\,C_R/L}{1-(\pi\Delta t\,C_R/L)^2}\right], \quad (34)$$

and the difference between this expression and C_R may then be attributed to the time truncation error alone.[2] In other words, the limiting speeds given by (34), and shown in figure 3, are the most accurate which could be found by reduction of the space mesh alone, and are a reasonable basis for comparison with the results of the extrapolation technique. In the case of $\Delta x_1=100$ km., for example, we see that as L increases an increasingly large fraction of the total possible phase speed improvement is provided by the present extrapolation. We also note that for $L>1500$ km. (with the assumed values of Δt, U, and β) the limiting phase speed need not be distinguished from C_R; this is to say that the time truncation affects mainly the shorter waves.

4. THE EXTRAPOLATION OF SOLUTIONS OF THE BAROTROPIC VORTICITY EQUATION

In the earlier discussion of the extrapolation technique, we noted that the method may evidently be applied either in the estimation of the individual differentials or in the improvement of the separate solutions themselves. The first method was considered in section 3 above, and we now turn our attention to the second approach, which may be termed "extrapolation of solutions." This technique is, in fact, that originally suggested by Richardson [6], who termed it the "deferred approach to the limit."

Continuing to use implicit differences for convenience, the difference equation approximating (15) is

[2] This same limiting expression also results from the unextrapolated solutions given by Gates [2].

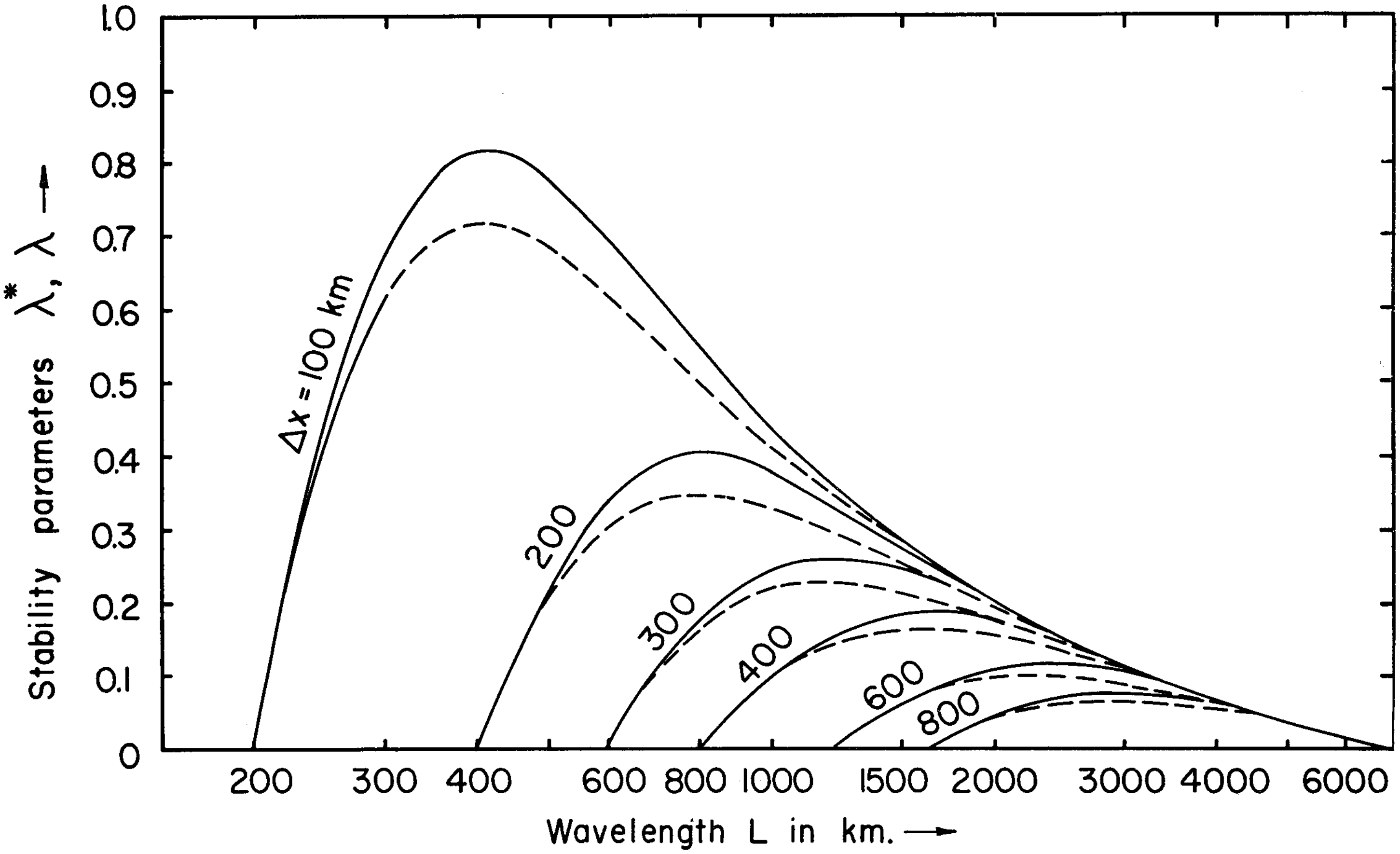

FIGURE 2.—The variation of the parameter λ^* of (28) with Δx and L. Also shown by the dashed lines is the corresponding parameter (29) for the unextrapolated difference case. Here $\Delta t=1$ hr., $U=20$ m.sec.$^{-1}$, and β (45°)$=1.619\times10^{-13}$ cm.$^{-1}$ sec.$^{-1}$

$$(\Delta t\Delta x^2)^{-1}(\psi_{m+1,\tau+1}-2\psi_{m,\tau+1}+\psi_{m-1,\tau+1}-\psi_{m+1,\tau}+2\psi_{m,\tau}-\psi_{m-1,\tau})+U(4\Delta x^3)^{-1}(\psi_{m+2,\tau}-2\psi_{m+1,\tau}+2\psi_{m-1,\tau}-\psi_{m-2,\tau}+\psi_{m+2,\tau+1}-2\psi_{m+1,\tau+1}+2\psi_{m-1,\tau+1}-\psi_{m-2,\tau+1})+\beta(4\Delta x)^{-1}(\psi_{m+1,\tau}-\psi_{m-1,\tau}+\psi_{m+1,\tau+1}-\psi_{m-1,\tau+1})=0. \tag{35}$$

The solution of this equation for the simple harmonic initial conditions used earlier is readily shown to be

$$\psi_{m,\tau}=\Psi \exp\left\{i\left[m\alpha-\tau\tan^{-1}\left(\frac{4\lambda}{4-\lambda^2}\right)\right]\right\}, \tag{36}$$

where λ is given by (29) and $\alpha=2\pi\Delta x/L$ as before.

We may now write this solution specifically for our two mesh sizes Δx_1 and Δx_2 as

$$\psi_{m_1,\tau}=\Psi \exp\left\{i\left[m_1\alpha_1-\tau\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right)\right]\right\}, \tag{37}$$

and

$$\psi_{m_2,\tau}=\Psi \exp\left\{i\left[m_2\alpha_2-\tau\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right)\right]\right\}, \tag{38}$$

and seek to combine them by the extrapolation technique. Since the truncation error of both (37) and (38) is $O(\Delta x^2)$, the appropriate extrapolation is of the form

$$\psi'_{m,\tau}\equiv(\rho^2\psi_{m_1,\tau}-\psi_{m_2,\tau})(\rho^2-1)^{-1}. \tag{39}$$

Selecting $\rho=\Delta x_2/\Delta x_1=2$ for convenience, we then have

$$\psi'_{m,\tau}=\frac{4}{3}\psi_{m_1,\tau}-\frac{1}{3}\psi_{m_2,\tau}. \tag{40}$$

Combining the separate solutions (37) and (38) according to (40), and noting $m_1\alpha_1=m_2\alpha_2$ at the points of the larger mesh, we find after some manipulation

$$\psi'_{m_2,\tau}=\Psi A\exp\{ik[m_2\Delta x_2-\tau\Delta tC'_N]\}, \tag{41}$$

where the amplitude factor A is given by

$$A=\frac{1}{3}\left\{17-8\cos\left[\tau\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1{}^2}\right)-\tau\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2{}^2}\right)\right]\right\} \tag{42}$$

and the numerical phase speed C'_N is given by

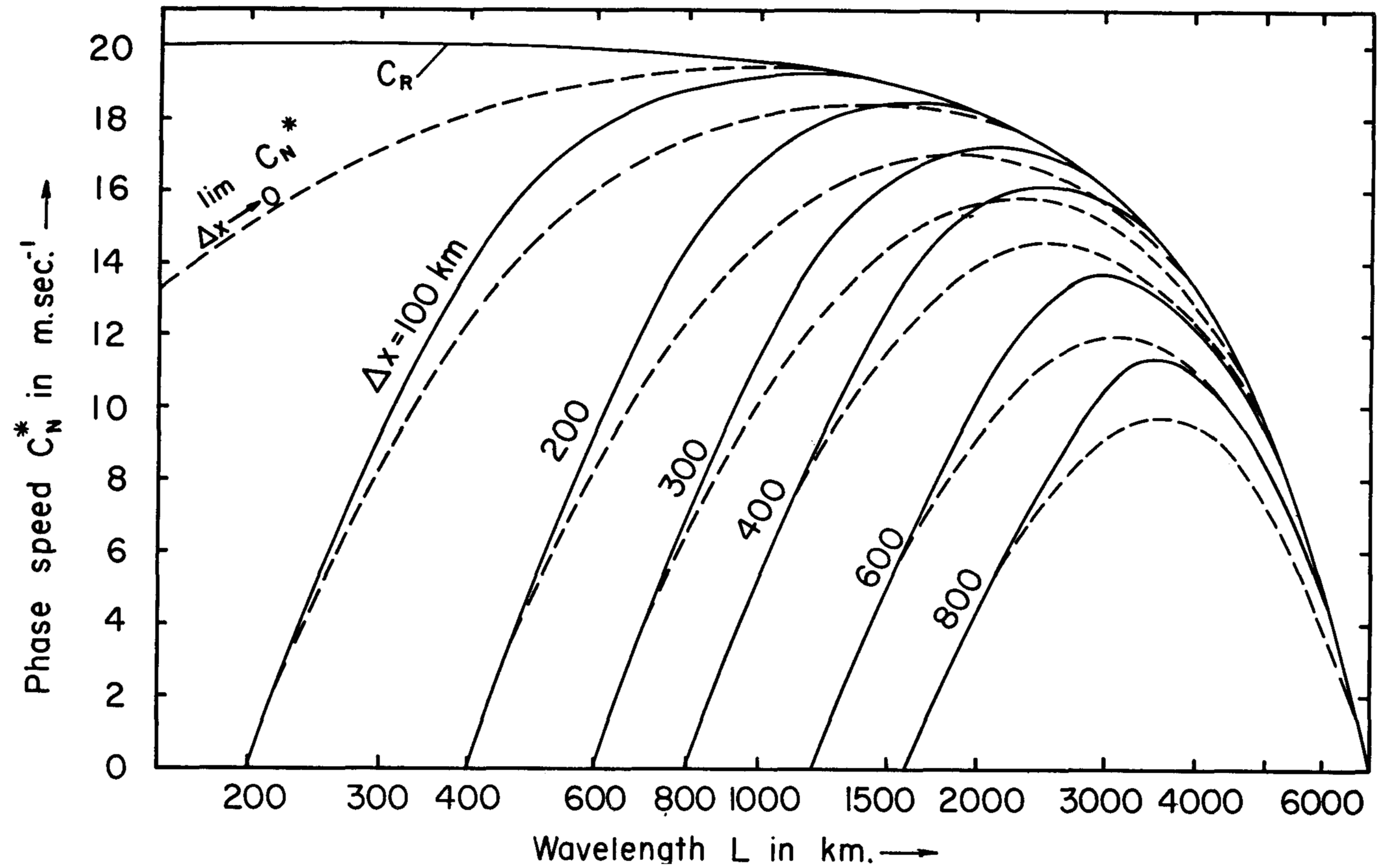

FIGURE 3.—The variation of the numerical phase speed C_N^* of (31) with Δx and L, for the "extrapolation of differences" technique. Also shown by the dashed lines is the corresponding phase speed for the unextrapolated difference case. The Rossby phase speed C_R of the continuous solution, and the limiting phase speed as $\Delta x \rightarrow 0$ are also given for comparison. Here $\Delta t=1$ hr., $U=20$ m. sec.$^{-1}$, and β (45°) $=1.619\times 10^{-13}$ cm.$^{-1}$ sec.$^{-1}$

$$C'_N=\frac{L}{2\pi\tau\Delta t}\tan^{-1}\left\{\frac{4\sin\left[\tau\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right)\right]-\sin\left[\tau\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right)\right]}{4\cos\left[\tau\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right)\right]-\cos\left[\tau\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right)\right]}\right\}. \tag{43}$$

In order to simplify the further analysis of these results, let us assume that the extrapolation of the two solutions (40) is performed at each time step. It then suffices to examine the solution (41) for $\tau=1$, and the amplitude factor and phase speed are accordingly simplified. If this is not done, and the extrapolation is considered to be applied after an arbitrary number of time steps τ, the amplitude A will vary with the choice of τ between the limits $1\leq A\leq 5/3$.[3] With $\tau=1$, the variation of A with Δx_1 and L is shown in table 1. We note that an amplitude distortion of a few percent of the extrapolated solution

[3] From the data presented by Gates [2], we may estimate that when $L=1000$ km., $\Delta x_1=200$ km., $\Delta x_2=400$ km., $\Delta t=1$ hr., $U=20$ m. sec.$^{-1}$, and $\beta(45°)=1.619\times 10^{-13}$ cm.$^{-1}$ sec.$^{-1}$ the maximum amplitude (5/3) would first occur for $\tau\simeq 14$.

TABLE 1.—*The variation of the $\tau=1$ amplitude distortion A of (42) as a function of wavelength L and selected mesh size Δx_1. Here the second mesh $\Delta x_2=2\Delta x_1$, $U=20$ m. sec.$^{-1}$, $\Delta t=1$ hr., and $\beta(45°)=1.619\times 10^{-13}$ cm.$^{-1}$ sec.$^{-1}$*

L (km.)	Grid mesh Δx_1, in km.			
	100	200	300	400
400	1.0966	----------	----------	----------
600	1.0189	----------	----------	----------
800	1.0044	1.0267	----------	----------
1000	1.0013	1.0114	*1.0117	----------
1500	1.0001	1.0016	1.0050	†1.0062
2000	1.0000	1.0003	1.0013	1.0027
2500	1.0000	1.0001	1.0004	1.0009
3000	1.0000	1.0000	1.0001	1.0003
4000	1.0000	1.0000	1.0000	1.0001
6000	1.0000	1.0000	1.0000	1.0000

*for $L=1200$ km.
†for $L=1600$ km.

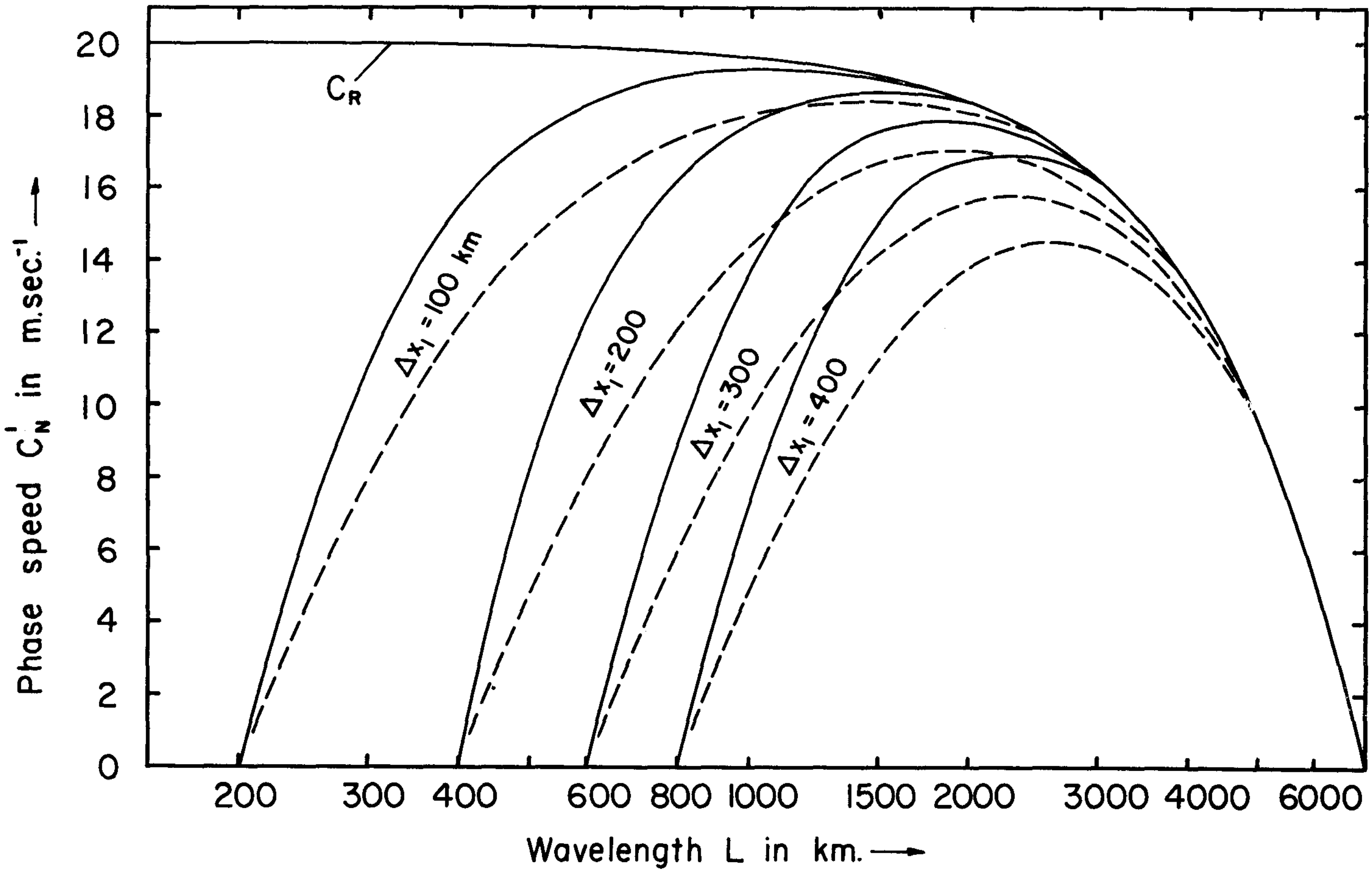

FIGURE 4.—The variation of the numerical phase speed C'_N of (43) with Δx_1 and L, for the "extrapolation of solutions" technique. Also shown by the dashed lines is the corresponding phase speed for the unextrapolated case. The Rossby phase speed C_R of the continuous solution is also given for comparison. Here $\Delta t = 1$ hr., $U = 20$ m. sec.$^{-1}$, and $\beta(45°) = 1.619 \times 10^{-13}$ cm.$^{-1}$ sec.$^{-1}$

(41) is confined to the shorter waves, for which the larger truncation errors are also made. For the longer wavelengths, $\lambda_1 \rightarrow \lambda_2$ and $A \rightarrow 1$.

The variation of the phase speed (43) of the extrapolated solution is shown in figure 4 for $\tau = 1$ as above. The phase speed improvement is seen to be most pronounced for the shorter waves; by comparison with figure 3 we note that approximately twice as much improvement over the unextrapolated solution is here obtained as was obtained by the technique of "extrapolation of differentials." For $\Delta x_1 = 100$ km., waves of length ≥ 1000 km. are now moved with very nearly the correct speed. The present method of "extrapolation of solutions" is therefore somewhat more effective than the method of section 3, at least as far as phase speeds are concerned.

5. A METHOD OF "OPTIMUM" EXTRAPOLATION

While the solution extrapolation technique described in the preceding section gives an overall improvement in phase speed, the small amplitude distortion also introduced (table 1) may be an undesirable feature for some purposes and accordingly may limit the practical application of the method. The solution extrapolation of (39) is based upon the elimination of the dominant spatial truncation error $O(\Delta x^2)$, and is in this sense only an approximation. It is possible to "extrapolate" the two solutions (37) and (38) in a slightly different way, and thereby to improve certain properties of the combined solution.

Let us form a simple linear combination of the solutions (37) and (38),

$$\psi''_{m,\tau} \equiv a\psi_{m_1,\tau} - b\psi_{m_2,\tau}, \tag{44}$$

where a and b are constants to be determined by the imposition of two conditions upon the solution $\psi''_{m,\tau}$. As a first condition, let us require the amplitude of $\psi''_{m,\tau}$ to equal Ψ, the amplitude of the initial conditions. This condition will then remove the amplitude factor A found in the previous method (at least for a certain wavelength, as discussed below). As a second condition, we may require that the solution $\psi''_{m,\tau}$ move at the phase speed

$$C''_N=(\rho^2 C_{N,1}-C_{N,2})(\rho^2-1)^{-1}, \tag{45}$$

where $C_{N,1}$ and $C_{N,2}$ are the phase speeds of the solutions (37) and (38). This condition is suggested by the more formal extrapolation technique, and insures that the solution $\psi''_{m,\tau}$ will move at a more accurate speed than does either of its component solutions. As before, let us select $\rho=2$, and note $m_1\alpha_1=m_2\alpha_2$. From these conditions we may then write (44) in the form

$$\begin{aligned}\psi''_{m,\tau}&=\Psi\exp\left\{i\left[m_2\alpha_2-\frac{4}{3}\tau\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right)\right.\right.\\&\qquad\left.\left.+\frac{1}{3}\tau\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right)\right]\right\}\\&=a\Psi\exp\left\{i\left[m_1\alpha_1-\tau\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right)\right]\right\}\\&\qquad-b\Psi\exp\left\{i\left[m_2\alpha_2-\tau\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right)\right]\right\},\end{aligned} \tag{46}$$

and thereby determine a and b. Setting $\tau=1$ as before and introducing the notation

$$\theta_1=\frac{1}{3}\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right), \tag{47}$$

$$\theta_2=\frac{1}{3}\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right) \tag{48}$$

for convenience, we find upon equating real and imaginary parts of each side of (46),

$$a=\frac{\sin[4(\theta_1-\theta_2)]}{\sin[3(\theta_1-\theta_2)]}, \tag{49}$$

$$b=\frac{\sin[\theta_1-\theta_2]}{\sin[3(\theta_1-\theta_2)]}. \tag{50}$$

For a selected Δx_1 and $\Delta x_2(=2\Delta x_1)$, the values of λ_1 and λ_2 vary with the wavelength L (see fig. 1). Hence θ_1 and θ_2 are likewise wavelength dependent, and it would appear that L must be specified for a unique determination of a and b. Fortunately, however, the values of a and b are not very sensitive to the value of L selected, as shown in table 2 below for the case $\Delta x_1=100$ km. For the longer wavelengths we notice that $a\rightarrow\frac{4}{3}$ and $b\rightarrow\frac{1}{3}$, corresponding to the extrapolation technique considered earlier in section 4. Since it is the shorter waves' phase speeds which require the most improvement, we may select $L=400$ km., the smallest permissible value (the resolution of the Δx_2 mesh). In this case, the phase speed of the solution $\psi''_{m,\tau}$,

$$C''_N=\frac{L}{2\pi\Delta t}\left[\frac{4}{3}\tan^{-1}\left(\frac{4\lambda_1}{4-\lambda_1^2}\right)-\frac{1}{3}\tan^{-1}\left(\frac{4\lambda_2}{4-\lambda_2^2}\right)\right], \tag{51}$$

varies as shown in figure 5. Here the corresponding phase speeds for the methods of sections 3 and 4, as well as that for the unextrapolated solution, are shown for comparison. We note that the use of $a=1.252$ and $b=0.358$ in (44) has given a small improvement over the use of $a=4/3$, and $b=1/3$ as in (40), which in turn is somewhat superior to the "extrapolation of differentials" of (27). This selection of a and b has, in effect, "tuned" the weighting scheme to give the greatest improvement at $L=400$ km., and is in this sense an "optimum" scheme. For other Δx_1, and a corresponding selection of a and b from table 2, generally similar results are obtained.

With this "tuning" to $L=400$ km., the amplitude is unity for this wavelength but is generally not unity for other wavelengths, as shown in table 3. This amplitude distortion, especially for the longer waves, greatly reduces the attractiveness of this "optimum" method, and is far more serious than that of the simpler method of "extrapolation of solutions" (39). The small phase speed improvement given by this method over that of (39) would therefore not be a useful improvement in practical application.

6. THE EXTRAPOLATION TECHNIQUE AND SMOOTHING

The basic extrapolation scheme (39) attempts to improve the accuracy of the finite-difference solutions by the removal of some of the spatial truncation error. Smoothing procedures (Shuman [9]), on the other hand, are designed to suppress the shorter waves (for which the larger truncation error is made) by the deliberate introduction of additional truncation, and in this sense the two procedures are opposed. If we write (39) for a discrete variable φ_m in the form

$$\varphi_{m_2}-\rho^2\varphi_{m_1}=(1-\rho^2)\varphi'_m, \tag{52}$$

where the notations are as before, and write Shuman's "three-point" smoothing operator for the same variable

Table 2.—*The variation of the weighting coefficients a and b of (44) with wavelength, for the case $\Delta x_1=100$ km., $U=20$ m. sec.$^{-1}$, $\rho=2$, $\Delta t=1$ hr., β (45°)$=1.619\times10^{-13}$ cm.$^{-1}$ sec.$^{-1}$*

	Wavelength L (km.)							
	400	600	800	1000	1500	2000	3000	6000
a	1.252	1.318	1.330	1.332	1.333	1.333	1.333	1.333
b	0.358	0.338	0.334	0.333	0.333	0.333	0.333	0.333

Table 3.—*The variation of the amplitude with wavelength of the extrapolated solution (46), for the choice $L=400$ km. in order to determine a (=1.252) and b (=0.358) in (49), (50). Here $\Delta x_1=100$ km., and ρ, U, Δt, and β are as in table 2.*

Wavelength L (km.)								
400	600	800	1000	1500	2000	3000	4000	6000
1.000	0.915	0.899	0.895	0.894	0.894	0.894	0.894	0.894

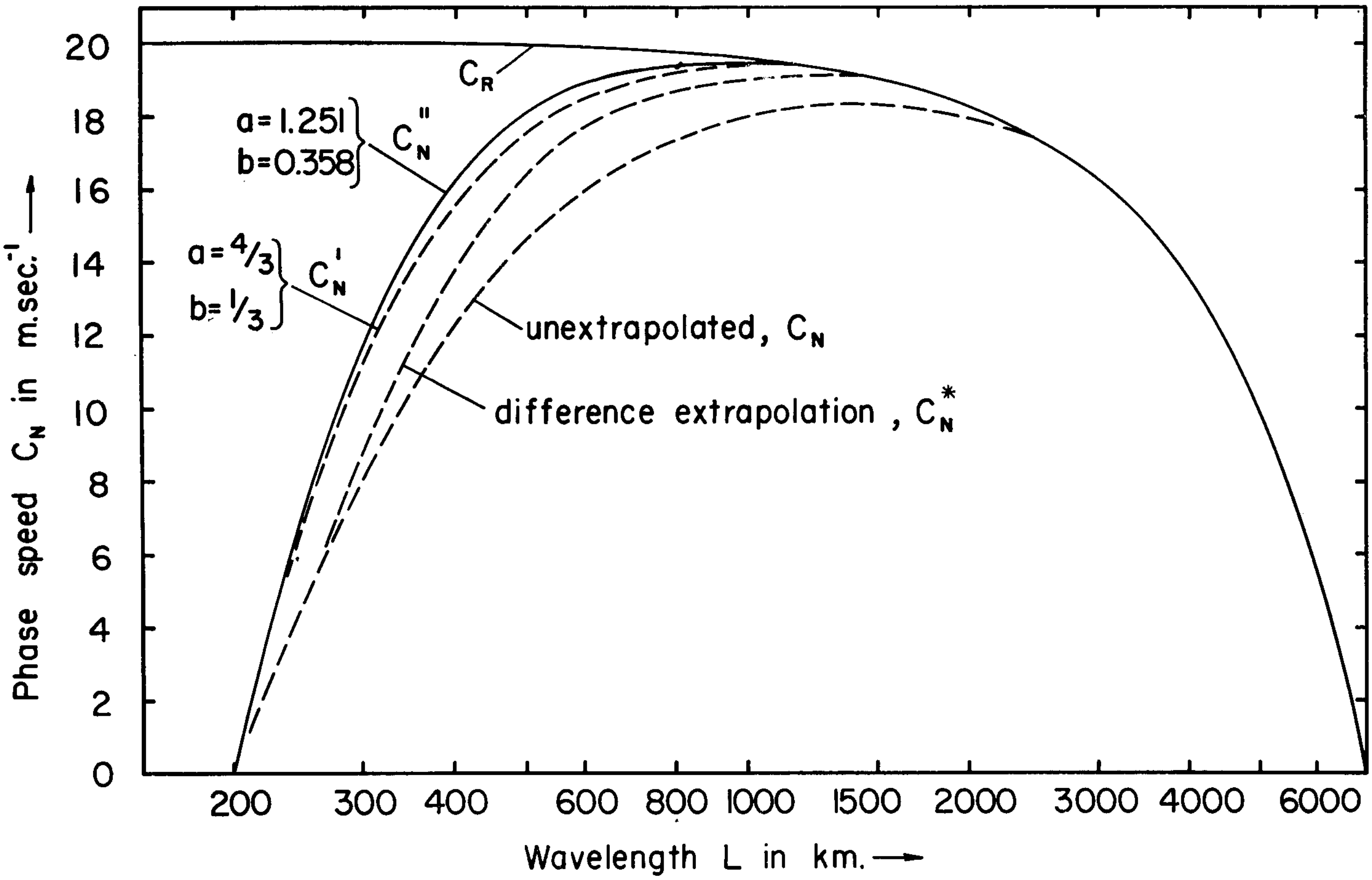

FIGURE 5.—The variation of the numerical phase speed C''_N of (51) with L for the "optimum" solution extrapolation at $L=400$ km., for the case $\Delta x_1=100$ km. Also shown for comparison are the corresponding phase speeds C'_N (43) for the usual solution extrapolation technique, C^*_N (31) for the difference extrapolation technique, C_N for the unextrapolated solution, and C_R (32), the Rossby (continuous) phase speed. Here $\Delta t=1$ hr., $U=20$ m. sec.$^{-1}$, and $\beta(45°)=1.619\times10^{-13}$ cm.$^{-1}$ sec.$^{-1}$

in the form

$$\varphi_m^{i+1}-\mu\varphi_m^i=(1-\mu)\left(\frac{\varphi_{m+1}^i+\varphi_{m-1}^i}{2}\right), \tag{53}$$

we may examine this relationship in more detail.

In (53), the superscript i denotes an "unsmoothed" value, and $i+1$ denotes a (once) "smoothed" value, while μ is the "smoothing index". We notice that on a formal basis the larger-mesh solution φ_{m2} corresponds to the local smoothed variable of Shuman, the smaller-mesh solution φ_{m1} corresponds to the local unsmoothed value, and the extrapolated solution φ'_m corresponds to the local space average of the unsmoothed variable of Shuman. With this interpretation the two procedures complement each other, and might well be employed together. For example, one could apply an extrapolation technique and *then* apply a smoothing operation, with the result that the behavior of those wavelengths retained by the smoothing would be better than if smoothing alone were applied. With $\Delta x_1=100$ km., say, and $\rho=2$, this could be accomplished with (40) and a smoothing operator of the Shuman type designed to suppress waves of length$\leq$600 km.

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7. CONCLUSIONS

The technique of extrapolating finite-difference estimates to effective zero mesh size systematically reduces the truncation error of solutions of the linear barotropic vorticity difference equation. This technique increases the numerical phase speeds toward the continuous solution's speeds, especially for the shorter waves, while preserving the solution's amplitude. The extrapolation of entire difference solutions found with two different mesh sizes results in an even greater phase speed improvement, and may be adjusted to give optimum results in the vicinity of selected wavelengths. While this "extrapolation of solution" technique improves the solution's phase speed, it introduces an amplitude distortion of certain wavelengths. The ordinary solution extrapolation method introduces about a 10-percent amplitude increase for the shorter waves only, while the "optimum" method introduces about a 10-percent amplitude decrease for *all* of the longer waves, and is consequently unsuitable for routine use. The method of "extrapolation by differentials," formally equivalent to the use of the higher-order difference approximations, may then be the best that can be expected from the extrapolation techniques. Whether the increased computation required by even this method is justified by the increased accuracy must await an actual numerical integration. Such a test is particularly important in view of the general observation that higher-order difference schemes often fail to provide the accuracy expected (Richtmyer [7]).

Although the implicit difference scheme has been employed in this analysis for convenience, the techniques described are in no way restricted to it. Noting that the numerical phase speeds of solutions found with the implicit difference scheme are less accurate than those found, say, with the first-forward-then-centered scheme (Gates [2]), an extrapolation technique with the latter difference scheme could be expected to give correspondingly more accurate results.

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